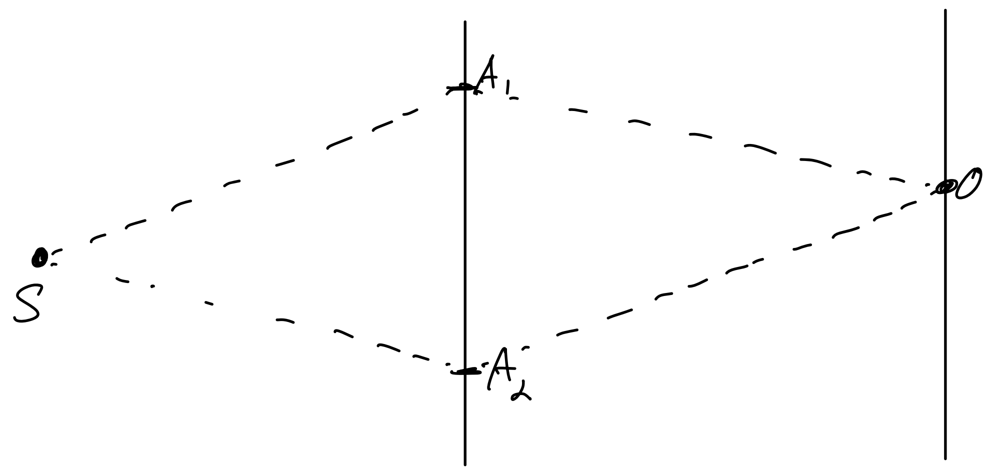
Introduction to Quntum Field Theory

1) What is Quantum Field Theory ?
2) How can we compute things in QFT ?
3) What general properties does the theory have and why is it important for mathematics ?
4) What does it teach us about the universe ?

1) → QFT arises from the marriage of "special relativity" with "quantum mechanics"
2) → Path integral formalism
3) → symmetries, conservation laws and role of topology (solitons, instantons, etc.)
4) → Standard Model and Quantum Gravity

§1. Path integral formulation of Quantum Physics

the double-slit experiment:



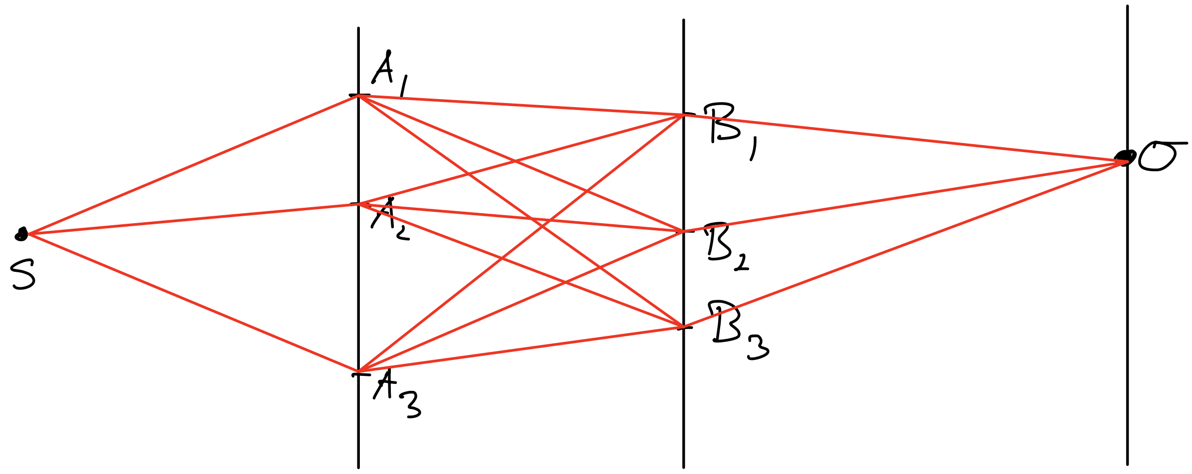
$\underline{t=0}$: particle emitted from source S

$\underline{t=T}$: detected at location $\mathcal{O}$

fundamental postulate of QM: amplitude at $\mathcal{O}$ is given by "superposition principle", namely as sum of amplitudes for propagation through A_1 and A_2.

Generalization:

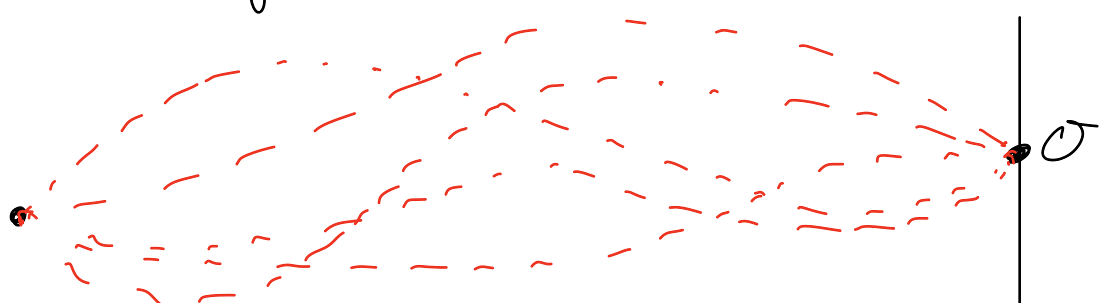
What if we have many screens and many holes in each screen?



The amplitude at detector O is given by:

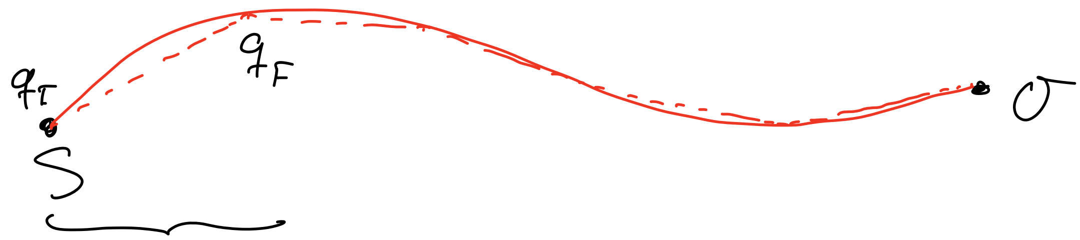
$$\mathcal{A}(\text{detected at } O) = \sum_{i,j} \mathcal{A}(S \longrightarrow A_i \longrightarrow B_j \longrightarrow O)$$

Continuing this logic, it follows that <u>without</u> any screens the amplitude is:



$$\mathcal{A}(S \rightarrow O \text{ in time } T) = \sum_{\text{(paths)}} \mathcal{A}(S \xrightarrow{\text{path}} O, T)$$

A continuous path can be subdivided into many line segments:



$$\to \mathcal{A}(q_I \to q_F) = \langle q_F | e^{-iH\Delta t} | q_I \rangle$$

$\to$ <u>Dirac's formulation</u>:

Divide T into N segments $\delta t = T/N$

$$\to \langle q_F | e^{-iHT} | q_I \rangle = \langle q_F | e^{-iH\delta t} e^{-iH\delta t} \cdots e^{-iH\delta t} | q_I \rangle$$

use complete set of states $\int dq \, |q\rangle\langle q| = \mathbb{1}$

$$\to \langle q_F | e^{-iHT} | q_I \rangle$$

$$= \left(\prod_{j=1}^{N-1} \int dq_j \right) \langle q_F | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \cdots \langle q_2 | e^{-iH\delta t} | q_1 \rangle \langle q_1 | e^{-iH\delta t} | q_I \rangle \qquad (*)$$

Let us take the free-particle Hamiltonian: $H = \hat{p}^2/2m$

recall $\hat{p}|p\rangle = p|p\rangle$ and $\langle q|p\rangle = e^{ipq}$

with normalization $\int \frac{dp}{2\pi} |p\rangle\langle p| = \mathbb{1}$

$$\rightarrow \langle q_{j+1}| e^{-i\delta t(\hat{p}^2/2m)} | q_j\rangle$$

$$= \int \frac{dp}{2\pi} \langle q_{j+1}| e^{-i\delta t(\hat{p}^2/2m)} |p\rangle\langle p| q_j\rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\delta t(p^2/2m)} \langle q_{j+1}|p\rangle\langle p| q_j\rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\delta t(p^2/2m)} e^{ip(q_{j+1}-q_j)}$$

Doing the integral over p, we get

$$= \left(\frac{-i2\pi m}{\delta t}\right)^{\frac{1}{2}} e^{im(q_{j+1}-q_j)^2/2\delta t}$$

$$= \left(-\frac{i2\pi m}{\delta t}\right)^{\frac{1}{2}} e^{i\delta t(m/2)\left((q_{j+1}-q_j)/\delta t\right)^2}$$

Inserting into (*) gives

$$\langle q_F| e^{-iHT} |q_{\underline{I}}\rangle$$

$$= \left(\frac{-i2\pi m}{\delta t}\right)^{\frac{N}{2}} \prod_{j=0}^{N-1} \int dq_j \, e^{i\delta t(m/2)\sum_{j=0}^{N-1}\left((q_{j+1}-q_j)/\delta t\right)^2}$$

with $q_0 := q_I$, $q_N := q_F$

Take continuum limit $\delta t \to 0$

$\to$ replace $[(q_{j+1} - q_j)/\delta t]^2$ by $\dot{q}^2$

$$\delta t \sum_{j=0}^{N-1} \quad \text{by} \quad \int_0^T dt$$

Define integral over paths

$$\int \mathcal{D}q(t) = \lim_{N \to \infty} \left(-\frac{i 2\pi m}{\delta t} \right)^{\frac{N}{2}} \prod_{j=0}^{N-1} \int dq_j$$

$\to$ path integral representation

$$\langle q_F | e^{-iHT} | q_I \rangle = \int \mathcal{D}q(t) \, e^{i \int_0^T dt \, \frac{1}{2} m \dot{q}^2}$$

For particle in a potential we get

$$H = \hat{p}^2/2m + V(\hat{q})$$

$$\to \langle q_F | e^{-iHT} | q_I \rangle = \int \mathcal{D}q(t) \, e^{i \int_0^T dt \, \underbrace{\left[\frac{1}{2} m \dot{q}^2 - V(q) \right]}_{= \mathcal{L}(\dot{q}, q)}}$$

(* *) "Lagrangian"

More generally, we say that the particle starts in some initial state I and ends in some final state F.

$$\rightarrow \int dq_F \int dq_I \, \langle F|q_F\rangle \langle q_F|e^{-iHT}|q_I\rangle \langle q_I|I\rangle$$

$$= \int dq_F \int dq_I \, \Psi_F(q_F)^* \langle q_F|e^{-iHT}|q_I\rangle \Psi_I(q_I)$$

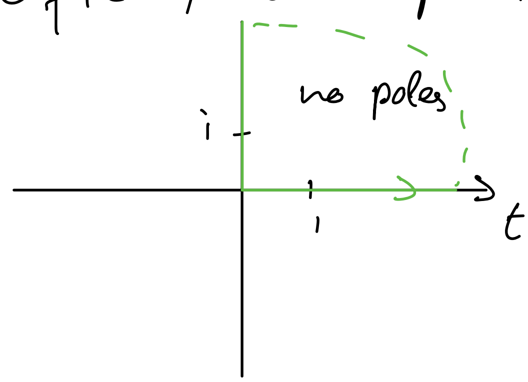
in most cases $I = F = |0\rangle$

ground state

and we denote

$$\langle 0|e^{-iHT}|0\rangle =: Z \quad \left(\begin{matrix}\text{partition}\\ \text{function}\end{matrix}\right)$$

Often, one performs "Wick rotation"



$\longrightarrow$ replace $t \to -it$

then we get

$$Z = \int \mathcal{D}q(t)\, e^{-\int_0^T dt \left[\frac{1}{2} m \dot{q}^2 + V(q)\right]}$$

"Euclidean path integral"

Classical limit of QM:

Restore Planck's constant $\hbar$ in (**):

$$\langle q_F | e^{-(i/\hbar)HT} | q_I \rangle = \int \mathcal{D}q(t)\, e^{(i/\hbar)\int_0^T dt\, \mathcal{L}(\dot{q}, q)}$$

Taking $\hbar \to 0$ in $I = \int_{-\infty}^{\infty} dq\, e^{-(1/\hbar) f(q)}$

and expanding around $q = a$, $f'(a) = 0$

$$f(q) = f(a) + \frac{1}{2} f''(a)(q-a)^2 + \mathcal{O}[(q-a)^3]$$

gives

$$I = e^{-(1/\hbar) f(a)} \left(\frac{2\pi\hbar}{f''(a)} \right)^{\frac{1}{2}} e^{-\mathcal{O}(\hbar^{\frac{1}{2}})}$$

applied to our case this gives

$$e^{(i/\hbar)\int_0^T dt\, \mathcal{L}(\dot{q}_c, q_c)},$$

where $q_c(t)$ is the "classical path" satisfying the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{q}} \right) - \left(\frac{\delta \mathcal{L}}{\delta q} \right) = 0$$